MATH 2410Q: Elementary Differential Equations
Spring 2019
A Brief Introduction to Linear Algebra

Name: Due date: February 14, 2019

As we progress through this course, we will consider differential equations in increasingly advanced settings. In particular, we will eventually develop tools that allow us to find solutions to systems of differential equations. The techniques we use to find solutions to systems of differential equations will require some basic knowledge of linear algebra. If you have taken MATH 2210Q (Applied Linear Algebra) then what follows will primarily be a review. However, a course in linear algebra is not a prerequisite for this course, and these notes and videos are intended to give a comprehensive overview of the linear algebra skills that we will use in this course.

This completed packet is due on Thursday February 14th and worth 5% of your overall grade. Please write your answers in the space provided within the packet. If you need additional space, you may write on the back side of the pages (indicate if you do so). You are encouraged to discuss the material and exercises in this packet with classmates. However, your final answers should be written independently. If you have questions about any of the material in this packet, you may ask during office hours or post to Piazza.

These notes follow closely along with sections B.1-B.3 in the course textbook. You may reference these sections for additional examples. Some topics from the textbook sections have been omitted since we will not need them in our course.

Unit 1: Matrix Basics

When performing mathematical computations, it is often beneficial to find a simple, condensed means of storing and referring to relevant data. Matrices (plural of matrix) are one tool that we can use to help us.

A **matrix** **A** is a rectangular array of numbers or functions

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The entry in the ith row and jth column of a matrix is denoted by a_{ij}

Exercise. Let

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 11 & 0 & -1 \\ 5 & -6 & 2 & 1 & 3 \\ 6 & 2 & 4 & 8 & 7 \end{bmatrix}$$

Example: $a_{15} = -1$.

1. (2 points) Identify the element a_{21}.

2. (2 points) Identify the element a_{32}.

3. (2 points) Identify the element a_{24}.

A matrix with m rows and n columns is said to have size $m \times n$ (read "m by n").

- If a matrix **A** has the same number of rows as columns, say **A** has size $n \times n$, then we call the matrix A a **square matrix** of order n.
- If a matrix **A** has m rows and 1 column then we call **A** a **column vector**.
- If a matrix **A** has 1 row and n columns then we call **A** a **row vector**.

REMINDER: Rows first then columns!

Exercise. For each of the matrices below, determine the size and then determine whether or not the matrix is a square matrix, a column vector, or a row vector.

Example: The matrix

$$\begin{bmatrix} 2 & 7 \\ 4 & 5 \\ 5 & 1 \end{bmatrix}$$

has size 3×2 and it is not a square matrix or a column vector or a row vector.

1. (4 points)

$$\begin{bmatrix} 1 & 9 & 7 & 4 \\ 5 & 2 & 8 & 1 \end{bmatrix}$$

Size:

This matrix is a

A. a square matrix.
B. a column vector.
C. a row vector.
D. none of the above

2. (4 points)

$$\begin{bmatrix} 5 & 7 & 8 & 9 \end{bmatrix}$$

Size:

This matrix is a

A. a square matrix.
B. a column vector.
C. a row vector.
D. none of the above

3. (4 points)

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Size:

This matrix is a

A. a square matrix.
B. a column vector.
C. a row vector.
D. none of the above

4. (4 points)

$$\begin{bmatrix} 5 & 8 & 3 \\ 7 & 2 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

Size:

This matrix is a

A. a square matrix.
B. a column vector.
C. a row vector.
D. none of the above

Let A and B be $m \times n$ matrices. We say A and B are **equal** if $a_{ij} = b_{ij}$ for each i and j.

$$\begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix} \text{ is NOT equal to } \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$$

because $a_{21} = 1$ but $b_{21} = 4$. For matrices to be equal, entries in corresponding positions need to agree!

Unit 2: Matrix Operations

In this unit will discuss three matrix operations: scalar multiplication, matrix addition, and matrix multiplication. We will also discuss determinants.

We define **scalar multiplication** on a matrix **A** by

$$k\mathbf{A} = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & & \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

where k is a number. That is, we multiply each entry in A by the number k.

Exercise. Perform the scalar multiplication indicated below.
Example:

$$5\begin{bmatrix} 2 & 1 \\ 4 & 9 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 20 & 45 \\ -10 & 15 \end{bmatrix}$$

1. (4 points) $6\begin{bmatrix} 0 & 2 & 1 \\ 4 & 2 & -4 \\ 3 & 1 & 5 \end{bmatrix}$

2. (4 points) $-2\begin{bmatrix} 1 & 4 & 3 & -2 \\ 4 & -1 & 0 & 0 \end{bmatrix}$

For this course, we may similarly multiply each entry in a matrix by a specified function.
Example:

$$e^x\begin{bmatrix} 2 & 1 \\ 4 & 9 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 2e^x & e^x \\ 4e^x & 9e^x \\ -2e^x & 3e^x \end{bmatrix}$$

Given two matrices of the same size, say $m \times n$, we define **matrix addition** by

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & & \\ b_{m1} & b_{m2} & ... & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & ... & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & ... & a_{2n}+b_{2n} \\ \vdots & \vdots & & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & ... & a_{mn}+b_{mn} \end{bmatrix}$$

That is, matrix addition is done entry-wise. Note that matrix addition between two matrices of different sizes is not defined.

Exercise. Compute the indicated sums.

Example:

$$\begin{bmatrix} 2 & 1 & 0 & -1 \\ 8 & 2 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 6 & 10 & 9 \\ -6 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 10 & 8 \\ 2 & 5 & 6 & 2 \end{bmatrix}$$

1. (4 points) $\begin{bmatrix} 2 & 4 & 1 \\ 3 & 2 & 1 \\ 4 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -2 \\ -3 & 8 & 7 \\ 5 & 5 & 1 \end{bmatrix}$

2. (4 points) $\begin{bmatrix} -3 & 2 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix}$

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ be a $n \times p$. Then the **product matrix AB** is the $m \times p$ matrix

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & ... & b_{1p} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & & \\ b_{n1} & b_{n2} & ... & b_{np} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+...+a_{1n}b_{n1} & ... & a_{11}b_{1p}+a_{12}b_{2p}+...+a_{1n}b_{np} \\ a_{21}b_{11}+a_{22}b_{21}+...+a_{2n}b_{n1} & ... & a_{21}b_{1p}+a_{22}b_{2p}+...+a_{2n}b_{np} \\ \vdots & \vdots & \vdots \\ a_{m1}b_{11}+a_{m2}b_{21}+...+a_{mn}b_{n1} & ... & a_{m1}b_{1p}+a_{m2}b_{2p}+...+a_{mn}b_{np} \end{bmatrix} \end{aligned}$$

In particular, if we want to find the element in the ith row and jth column of $\mathbf{AB}$, we use the entries from row i of matrix $\mathbf{A}$ and the entries from column j of matrix $\mathbf{B}$.

Exercise. Compute the indicated products

Example: $\begin{bmatrix} 2 & 6 \\ 4 & 7 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & 1 \\ 5 & 9 \end{bmatrix}.$

(See Unit 2 Video 1 for instructions)

1. (4 points) $\begin{bmatrix} 1 & 2 & 7 \\ 4 & 5 & 3 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 0 & 1 \\ 3 & 6 \end{bmatrix}$

2. (4 points) $\begin{bmatrix} -2 & 4 \\ 0 & 1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 7 \\ 4 & 5 & 3 \end{bmatrix}$

3. (4 points) $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix}$

4. (4 points) $\begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Note that the product **AB** is only defined when the number of columns of matrix **A** matches the number of row of matrix **B**. Exercises 1 and 2 demonstrate that matrix multiplication is not commutative. That is, in general $\mathbf{AB} \neq \mathbf{BA}$. However, associative and distributive laws do hold for matrix multiplication:

Associative Law: Let **A** be an $m \times n$ matrix, **B** be an $n \times p$ matrix, and **C** be a $p \times r$ matrix. Then

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

is an $m \times r$ matrix.

Distributive Law: If the products and addition are defined then

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \text{ and } (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}.$$

There are two special types of matrices: Zero matrices and identity matrices. A **zero matrix** is a matrix of any size consisting of all zero entries. Regardless of size (which is typically inferred by the surrounding information) the zero matrix is denoted by **0**. For example:

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

An **identity matrix** is a square matrix (of size $n \times n$) with 1's along in positions a_{ii}, $1 \leq i \leq n$ and 0's elsewhere. For example:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The $n \times n$ identity matrix is the multiplicative identity for $n \times n$ matrices. If **A** has size $n \times n$ then

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}.$$

Furthermore, if **X** is a $n \times 1$ column vector then

$$\mathbf{IX} = \mathbf{X}$$

(but **XI** is not defined).

To each matrix we associate a number called the **determinant**. For this course, you will only need to compute determinants of 2×2 and 3×3 matrices.

Given a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ the determinant of **A** is defined to be

$$\text{det}\mathbf{A} = a_{11}a_{22} - a_{21}a_{22}.$$

Given a 3×3 matrix $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ the determinant of **B** is defined to be

$$\text{det}\mathbf{B} = b_{11}\text{det}\left(\begin{bmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{bmatrix}\right) - b_{12}\text{det}\left(\begin{bmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{bmatrix}\right) + b_{13}\text{det}\left(\begin{bmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}\right).$$

Exercise. Find the determinant of each matrix.

Example:

$$\det\left(\begin{bmatrix} 4 & 7 \\ 2 & -1 \end{bmatrix}\right) = 4(-1) - 2(7)$$

Example:

$$\det\left(\begin{bmatrix} 2 & 0 & 4 \\ 1 & -3 & 3 \\ 6 & 8 & 5 \end{bmatrix}\right) = 26$$

(See Unit 2 Video 2 for instructions.)

1. (4 points) $\begin{bmatrix} 1 & 2 \\ -6 & 5 \end{bmatrix}$

2. (4 points) $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ -1 & 0 & 5 \end{bmatrix}$

Unit 3: Linear Systems and Gauss-Jordan Elimination

As mentioned in the introduction, matrices are an important tool for us to use to organize mathematical data in a concise way. In particular, we can store information about a system of m linear equations in n variables.

Consider the system of two equations and two variable (x and y).

$$\begin{cases} 2x + y = 7 \\ x + 3y = 5 \end{cases} . \tag{1}$$

Notice that

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x + 3y \end{bmatrix}.$$

Thus we can rewrite the system (1) in matrix form as

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

Instead of working with directly with the equations to find a solution (that is, quantities for x and y that make the equations in (1) true simultaneously), we can work with the **augmented matrix** that corresponds to the system

$$\left[\begin{array}{cc|c} 2 & 1 & 7 \\ 1 & 3 & 5 \end{array}\right].$$

We say an augmented matrix is in **reduced row echelon form** if the following properties hold:

(F1) The leading nonzero entry in a nonzero **row** is 1.

(F2) In consecutive nonzero **rows**, the leading entry in the lower row appears to the right of the leading entry in the higher row.

(F3) **Rows** consisting of all 0's are at the bottom of the matrix.

(F4) A **column** containing a leading 1 has 0's everywhere else.

Exercise. Indicate whether or not matrices below are in reduced row echelon form. If a matrix is not in reduced echelon form, indicate which of the properties is not satisfied.

Example:
The following matrices are in reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 8 \end{array}\right], \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 2 & 5 \end{array}\right]$$

The following matrix is not in reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 4 & 0 & 4 \end{array}\right]$$

In the above matrix property (F2) is not satisfied (see rows 2 and 3). Property (F1) is also not satisfied (see leading entry in row 3).

1. (2 points) $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -4 & 6 \end{array}\right]$

2. (2 points) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right]$

We use the following **elementary row operations** to transform the augmented matrix for our system into reduced row echelon form:

(E1) Multiply a row by a nonzero constant.

(E2) Interchange rows.

(E3) Add a nonzero multiple of one row to another row.

The elementary row operation are analogues to simple algebraic properties we use to solve systems directly in equation form. The process of converting a system to an augmented matrix and using elementary row operations to transform the matrix to reduced row echelon form is called **Gauss-Jordan Elimination**.

Exercise. Use Gauss-Jordan Elimination to solve the systems below. Show your work.

Example:

$$\begin{cases} 2x + y = 7 \\ x + 3y = 5 \end{cases}$$

(See Unit 3 Video 1 for instructions)

Solution: $(16/5, 3/5)$

Example:

$$\begin{cases} 3x + y + z = 4 \\ 4x + 2y - z = 7 \\ x + y - 3z = 6 \end{cases}$$

(See Unit 3 Video 2 for instructions)

Solution: $(5, -8, -3)$

1. (4 points) $\begin{cases} x + 4y = 6 \\ 2x + 3y = 9 \end{cases}$

2. (4 points) $\begin{cases} 3y + 3z = 0 \\ x - 3y = 5 \\ -2x + 2y + 10z = 4 \end{cases}$

For the systems above, the corresponding reduced row echelon form matrices have a leading 1 in each column to the left of the augmentation line. Whenever there is a leading 1 in each column to the left of the augmentation line, the system has exactly one (unique) solution. If some column of the reduced row echelon form matrix does not have a leading 1 then the system has infinitely many solutions.

Exercise. Suppose that the augmented matrices below correspond to systems of equations in the variables x, y, and z. Find all solutions to the systems. Show your work.

Example:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

(See Unit 3 Video 3 for instructions)
Solution: $(6-4t, 3-2t, t)$ where t is any real number.

1. (4 points) $\left[\begin{array}{ccc|c} 1 & 0 & 4 & -2 \\ 0 & 1 & 3 & -7 \end{array}\right]$

2. (4 points) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Unit 4: Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be an $n \times n$ square matrix. A number λ is an **eigenvalue** of $\mathbf{A}$ if there exists a nonzero (column) vector $\mathbf{X}$ that solves the matrix equation

$$\mathbf{AX} = \lambda\mathbf{X}.$$

If such a vector $\mathbf{X}$ (not all zero entries) exists then $\mathbf{X}$ is said to be an **eigenvector** corresponding to eigenvalue λ. For example, the vector $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}$ corresponding to eigenvalue $\lambda = 2$ since

$$\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

The equation $\mathbf{AX} = \lambda\mathbf{X}$ can be rewritten a $\mathbf{AX} - \lambda\mathbf{X} = 0$ or $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = 0$ where $\mathbf{I}$ is the $n \times n$ identity matrix. Of course the vector $\mathbf{X}$ containing all zero entries makes this equation true for any value of λ, but such vector does not qualify as an eigenvector. To find eigenvectors, we first find eigenvalues. To find the eigenvalues of $\mathbf{A}$, we solve

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0. \tag{2}$$

Equation (2) is a polynomial equation in λ called the **characteristic equation of A**. The solutions to the equation are the desired eigenvalues.

Exercise. Find all eigenvalues of the indicated matrix.

Example:

$$\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

(See Unit 4 Video 1 for instructions)

Solution: $\lambda = -1$, $\lambda = 4$, $\lambda = 2$.

1. (4 points) $\begin{bmatrix} 8 & 0 & 0 \\ 7 & -1 & -2 \\ -7 & 0 & 1 \end{bmatrix}$.

Once we have found an eigenvalue λ, we can find the corresponding eigenvector by solving $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}$ for $\mathbf{K}$. That is, we can apply Gauss-Jordan elimination to the augmented matrix $(\mathbf{A} - \lambda\mathbf{I}|\mathbf{0})$.

Exercise. Find eigenvectors of the for the indicated matrix and specified eigenvalue

Example:

$$\begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \lambda = 4$$

(See Unit 4 Video 2 for instructions)

Solution: $c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ where c is a real number.

1. (4 points) $\begin{bmatrix} 5 & 3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}, \lambda = 1$